Hyperbola

1. Prove that the point $P\left[\frac{a}{2}\left(p+\frac{1}{p}\right), \frac{b}{2}\left(p-\frac{1}{p}\right)\right]$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Another point on this hyperbola is given by $Q\left[\frac{a}{2}\left(q+\frac{1}{q}\right), \frac{b}{2}\left(q-\frac{1}{q}\right)\right]$.

Find the equation of chord PQ.

Deduce that the equation of the tangent to the hyperbola at P is given by

$bx(p^2+1)-ay(p^2-1)=2abp$.

This tangent intersects the x-axis at the point A and the y-axis at the point B.

Find the area of ΔOAB in terms of p.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{\left[\frac{a}{2}\left(p+\frac{1}{p}\right)\right]^2}{a^2}-\frac{\left[\frac{b}{2}\left(p-\frac{1}{p}\right)\right]^2}{b^2}=\frac{1}{4}\left[\left(p^2+2+\frac{1}{p^2}\right)-\left(p^2-2+\frac{1}{p^2}\right)\right]=1$$

Therefore the point $P\left[\frac{a}{2}\left(p+\frac{1}{p}\right), \frac{b}{2}\left(p-\frac{1}{p}\right)\right]$ lies on the curve $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Gradient of the chord PQ $=m=\frac{\frac{b}{2}\left(q-\frac{1}{q}\right)-\frac{b}{2}\left(p-\frac{1}{p}\right)}{\frac{a}{2}\left(q+\frac{1}{q}\right)-\frac{a}{2}\left(p+\frac{1}{p}\right)}=\frac{b(pq+1)}{a(pq-1)}$

The equation of chord PQ: $\mathbf{y-\frac{b}{2}\left(p-\frac{1}{p}\right)=\frac{b(pq+1)}{a(pq-1)}\left[x-\frac{a}{2}\left(p+\frac{1}{p}\right)\right]}$

The chord becomes the tangent to the curve at P when Q = P, that is, q = p.

$$y-\frac{b}{2}\left(p-\frac{1}{p}\right)=\frac{b(p^2+1)}{a(p^2-1)}\left[x-\frac{a}{2}\left(p+\frac{1}{p}\right)\right]$$

$$y-\frac{b}{2}\left(\frac{p^2-1}{p}\right)=\frac{b(p^2+1)}{a(p^2-1)}\left[x-\frac{a}{2}\left(\frac{p^2+1}{p}\right)\right]$$

$$ay(p^2-1)-\frac{ab}{2p}(p^2-1)^2=bx(p^2+1)-\frac{ab}{2p}(p^2+1)^2$$

$$bx(p^2+1)-ay(p^2-1)=\frac{ab}{2p}(p^2+1)^2-\frac{ab}{2p}(p^2-1)^2$$

$$\mathbf{bx(p^2+1)-ay(p^2-1)=2abp}$$

This tangent intersects the x-axis when $bx(p^2+1)-a(0)(p^2-1)=2abp$ or $x=\frac{2ap}{p^2+1}$

and intersects the y-axis when $b(0)(p^2+1)-ay(p^2-1)=2abp$ or $y=-\frac{2bp}{p^2-1}$

$A=\left(\frac{2ap}{p^2+1},0\right)$, $B\left(0,-\frac{2bp}{p^2-1}\right)$

$\therefore$ Area of $\Delta OAB=\frac{1}{2}\left(\frac{2ap}{p^2+1}\right)\left(\frac{2bp}{p^2-1}\right)=\mathbf{\frac{2abp^2}{(p^2-1)(p^2+1)}}$

2. Given four points $A(a,0), A'(-a,0), B(b,0), B'(-b,0)$, $a \neq b$. A point P moves so that its distances are related by the equation: $AP \cdot AP' = BP \cdot BP'$.

Show that the locus of P is a hyperbola and find the equations of its asymptotes.

Let $P(x,y)$,

$$AP \cdot AP' = BP \cdot BP'$$

$$AP^2 \cdot AP'^2 = BP^2 \cdot BP'^2$$

$$[(x-a)^2+y^2][(x+a)^2+y^2] = [(x-b)^2+y^2][(x+b)^2+y^2]$$

$$[(x^2-2ax+a^2)+y^2][(x^2+2ax+a^2)+y^2] = [(x^2-2bx+b^2)+y^2][(x^2+2bx+b^2)+y^2]$$

$$[(x^2+y^2+a^2)-2ax][(x^2+y^2+a^2)+2ax] = [(x^2+y^2+b^2)-2bx][(x^2+y^2+b^2)+2bx]$$

$$(x^2+y^2+a^2)^2-4a^2x^2 = (x^2+y^2+b^2)^2-4b^2x^2$$

$$(x^2+y^2)^2+2(x^2+y^2)a^2+a^4-4a^2x^2 = (x^2+y^2)^2+2(x^2+y^2)b^2+b^4-4b^2x^2$$

$$2(x^2+y^2)(a^2-b^2)+a^4-b^4-4x^2(a^2-b^2)=0$$

$$(a^2-b^2)[2(x^2+y^2)+(a^2+b^2)-4x^2]=0$$

$$2(x^2+y^2)+(a^2+b^2)-4x^2=0$$

The locus of P is a hyperbola: $y^2-x^2=\frac{a^2+b^2}{2}$

The asymptotes of the hyperbola are $y^2-x^2=0$ or $y=\pm x$.

3. **(a)** A curve of the form $\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1$ has asymptotes $y^2=m^2x^2$ and passes through the point $(a,0)$. Find the equation of the this curve in terms of x, y, a and m.

(b) A point P on this curve is equidistant from one of its asymptotes and the x-axis.

Prove that, for all values m, P lies on the curve : $(x^2-y^2)^2=4x^2(x^2-a^2)$

(a) The asymptotes of the hyperbola $\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1$ are $\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=0$ or $y^2=\frac{\beta^2}{\alpha^2}x^2$.

Compare this with the given symptotes $y^2=m^2x^2$, we have $m^2=\frac{\beta^2}{\alpha^2}$.

Hence, $\beta^2 = m^2\alpha^2$.

The hyperbola is therefore $\frac{x^2}{\alpha^2} - \frac{y^2}{m^2\alpha^2} = 1$ or $x^2 - \frac{y^2}{m^2} = \alpha^2$.

$(a, 0)$ is on the hyperbola, therefore $\frac{a^2}{\alpha^2} - \frac{0^2}{\beta^2} = 1$, we get $\alpha^2 = a^2$.

The required equation of the hyperbola is $\mathbf{x^2 - \frac{y^2}{m^2} = a^2}$.

(b) Let the point P on the hyperbola $x^2 - \frac{y^2}{m^2} = a^2$ be $P(x_0, y_0)$.

The asymptotes $y = \pm mx$ or $\pm mx - y = 0$.

The distance of P to the asymptotes is $\left|\frac{\pm mx_0 - y_0}{\sqrt{m^2+1}}\right|$.

The distance of P to x-axis is y_0. $\therefore \left|\frac{\pm mx_0 - y_0}{\sqrt{m^2+1}}\right| = y_0$

$${y_0}^2 = \left(\frac{\pm mx_0 - y_0}{\sqrt{m^2+1}}\right)^2$$

$$(m^2+1){y_0}^2 = m^2{x_0}^2 \pm 2mx_0y_0 + {y_0}^2$$

$$m^2({x_0}^2 - {y_0}^2) = \pm 2mx_0y_0$$

$$\pm m = \frac{2x_0y_0}{{x_0}^2 - {y_0}^2}$$

$$m^2 = \left(\frac{2x_0y_0}{{x_0}^2 - {y_0}^2}\right)^2 \ldots (1)$$

Since $P(x_0, y_0)$ is on $x^2 - \frac{y^2}{m^2} = a^2$, ${x_0}^2 - \frac{{y_0}^2}{m^2} = a^2 \ldots (2)$

Substitute (1) in (2), ${x_0}^2 - \frac{{y_0}^2}{\left(\frac{2x_0y_0}{{x_0}^2-{y_0}^2}\right)^2} = a^2$

$${x_0}^2 - a^2 = \frac{({x_0}^2 - {y_0}^2)^2}{4{x_0}^2}$$

$$({x_0}^2 - {y_0}^2)^2 = 4{x_0}^2({x_0}^2 - a^2)$$

Therefore $P(x_0, y_0)$ is on the curve $(x^2 - y^2)^2 = 4x^2(x^2 - a^2)$.

4. The tangents to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ at points A and B on the curve meet at point T.

If M is the mid-point of AB, prove that TM passes through the center of the hyperbola.
Prove that the product of the slopes of AB and TM is a constant.

Let $A\left[\frac{a}{2}\left(s+\frac{1}{s}\right), \frac{b}{2}\left(s-\frac{1}{s}\right)\right], B\left[\frac{a}{2}\left(t+\frac{1}{t}\right), \frac{b}{2}\left(t-\frac{1}{t}\right)\right]$ be two points on $b^2x^2 - a^2y^2 = a^2b^2$.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y} \Rightarrow \left.\frac{dy}{dx}\right|_A = \frac{b^2\left[\frac{a}{2}\left(s+\frac{1}{s}\right)\right]}{a^2\left[\frac{b}{2}\left(s-\frac{1}{s}\right)\right]} = \frac{b\,(s^2+1)}{a\,(s^2-1)}$$

Equation of tangent at A: $\quad y - \frac{b}{2}\left(s-\frac{1}{s}\right) = \frac{b\,(s^2+1)}{a\,(s^2-1)}\left[x - \frac{a}{2}\left(s+\frac{1}{s}\right)\right]$

$$bx(s^2+1) - ay(s^2-1) = 2abs \quad \ldots(1)$$

Equation of tangent at B:

$$bx(t^2+1) - ay(t^2-1) = 2abt \quad \ldots(2)$$

Solve (1) and (2), $x = \frac{a(st+1)}{s+t}, y = \frac{b(st-1)}{s+t}$. $\quad T\left(\frac{a(st+1)}{s+t}, \frac{b(st-1)}{s+t}\right)$

$$M = \left(\frac{\frac{a}{2}\left(s+\frac{1}{s}\right)+\frac{a}{2}\left(t+\frac{1}{t}\right)}{2}, \frac{\frac{b}{2}\left(s-\frac{1}{s}\right)+\frac{b}{2}\left(t-\frac{1}{t}\right)}{2}\right) = \left(\frac{a(s+t)(st+1)}{4st}, \frac{b(s+t)(st-1)}{4st}\right)$$

Let C(0,0) be the centre of the hyperbola,

Slope of TC $= \dfrac{\frac{b(st-1)}{s+t}}{\frac{a(st+1)}{s+t}} = \dfrac{b\,(s\,t-1)}{a\,(s\,t+1)}$, $\qquad$ Slope of MC $= \dfrac{\frac{b(s+t)(st-1)}{4st}}{\frac{a(s+t)(st+1)}{4st}} = \dfrac{b\,(s\,t-1)}{a\,(s\,t+1)}$

Hence, Slope of TC = Slope of MC and TMC is a straight line.

Therefore TM passes through the center of the hyperbola.

Slope of TM $= \dfrac{b\,(s\,t-1)}{a\,(s\,t+1)}$

Slope of AB $= \dfrac{\frac{b}{2}\left(s-\frac{1}{s}\right)-\frac{b}{2}\left(t-\frac{1}{t}\right)}{\frac{a}{2}\left(s+\frac{1}{s}\right)-\frac{a}{2}\left(t+\frac{1}{t}\right)} = \dfrac{b\,(st+1)}{a\,(st-1)}$

Slope of TM x Slope of AB $= \dfrac{b\,(s\,t-1)}{a\,(s\,t+1)}\dfrac{b\,(st+1)}{a\,(st-1)} = \dfrac{b^2}{a^2}$ (which is a constant).

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